Integrable deformations of systems on graphs with loops

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Studying spectral properties and scattering processes for the Schrödinger-type operators on infinite trees [1] [2] has provided many unexpected results including the relation between the scattering ($S-$) matrix on an infinite tree with cycles and Artin–Selberg L-function [3]. However, no reasonable integrable deformation of second-order operators that resembles that for the lattices $\mathbf{Z}$ ((L, A)-pairs of the Toda chain type) and $\mathbf{Z}_2$ ((L, A, B)-triples) (see [4]) had been constructed.

In [5], Krichever and Novikov had constructed the deformation of arbitrary fourth-order symmetrical operator L in the form of the (L, A, B)-triple, $\dot{L} = LA - BL$, for the T_3 graph (infinite homogeneous tree with vertices of valence three). This deformation preserves the spectral data of the zero level $L\psi = 0$.

In [6], the Krichever–Novikov integrable system was generalized to the case of an arbitrary (inhomogeneous) tree graph T. An arbitrary graph Γ with finite number of cycles can be innambiguously (up to isomorphisms) presented as the quotient of the universal covering tree graph T by the action of a finitely generated subgroup Δ of the total group of symmetries of the tree graph T, $\Gamma = T/\Delta$.

We consider operators L acting on the space of functions ψ_P on vertices of a graph Γ. The distance $d(X_1, X_2) \in \mathbf{Z}_+ \cap \{0\}$ on a tree graph is measured in number of edges entering the path connecting the vertices X_1 and X_2. The order of equation $L\psi = 0$ where $(L\psi)_P = \sum_X b_{P,X}\psi_X$ is the maximal diameter $\max_P d(X_1, X_2) : b_{P,X_1} \neq 0, b_{P<X_2} \neq 0$ or $b_{X_1,X_2} \neq 0$. In the general graph, paths connecting two vertices X_1 and X_2 are not unique. We can then consider the theory on the covering tree T claiming all functions ψ_X and coefficients of the operators L, A, and B to be periodic with respect to the action of the symmetry subgroup Δ. On the factorized graph Γ, this means that we associate coefficients of the operators with *paths* of the corresponding lengths connecting preimages $\tilde{X}_1, \tilde{X}_2 \in T$ of the corresponding points $X_1, X_2 \in \Gamma$.

We first consider an arbitrary graph without loops.

Theorem 1 *[6] Real self-adjoint operator L of the fourth order on a graph Γ without loops admits an isospectral deformation of a zero energy level $L\psi = 0$ having the form of the (L, A, B)-triple $[B = A^t]$,*

$$\dot{L} = LA + A^t L, \tag{1}$$

where A is the second-order operator,

$$(A\psi)_X = \sum_{X' :\, d(X,X')=1} a_{X,X'}\psi_{X'} + s_X\psi_X, \tag{2}$$

iff the operator L,

$$(L\psi)_X = \sum_{X'' :\, d(X,X'')=2} b_{X,X''}\psi_{X''} + \sum_{X' :\, d(X,X')=1} r_{X,X'}\psi_{X'} + \rho_X\psi_X \tag{3}$$

(such that all $b_{X,X''} > 0$) can be presented in the form

$$L = Q^t Q + u_X, \tag{4}$$

where Q is a second-order operator

$$(Q\psi)_X = \sum_{X' :\, d(X,X')=1} q_{X,X'}\psi_{X'} + v_X\psi_X. \tag{5}$$

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In the case where $b_{X,X''} \equiv b_{X'',X} > 0$, the coefficients $q_{X,X'}$ are inambiguously determined by the coefficients $b_{X,X''}$ and vice versa. In particular, for arbitrary three points X_1, X_2, X_3 such that $X_1 \neq X_3$ and $d(X_i, X_{i+1}) = 1$, i=1,2, $b_{X_1,X_3} = q_{X_1,X_2} q_{X_3,X_2}$. The coefficients v_X depend on a free parameter—the value of v_{X_0} at the given point X_0 (the center). [An arbitrary fourth-order operator L (3) on the tree Γ_3 admits representation (4) and, hence, isospectral deformation (1) [5].]

The coefficients $a_{X,X'}$ of deformation operator (2) are inambiguously expressed (up to a total multiplicative constant) via $q_{X,Y}$. For arbitrary three vertices

$$X_1, X_2, X_3 : \ d(X_i, X_{i+1}) = 1, \ i = 1, 2, \tag{6}$$

we obtain[1]

$$\frac{a_{X_2,X_1}}{q_{X_1,X_2}} = \frac{a_{X_2,X_3}}{q_{X_3,X_2}} \text{ and } a_{X_3,X_2} = -a_{X_2,X_1} \frac{q^2_{X_2,X_3}}{q_{X_1,X_2} q_{X_3,X_2}} \tag{7}$$

[in particular, when $X_1 = X_3$ we obtain $a_{X_1,X_2} = -a_{X_2,X_1} q^2_{X_2,X_1} / q^2_{X_1,X_2}$].

Let $\{X'_j\}$ and $\{X''_i\}$ be the sets of vertices that are neighbor to the respective vertices X_1 and X_2 but do not coincide with X_1 or X_2. Explicit formulas for the deformation read

$$\begin{aligned}
\dot b_{X_1,X_3} &= r_{X_1,X_2} a_{X_2,X_3} + a_{X_2,X_1} r_{X_2,X_3} + b_{X_1,X_3}(s_{X_1} + s_{X_3}) \\
\dot r_{X_1,X_2} &= \sum_{X''} b_{X_1,X''} a_{X'',X_2} + \sum_{X'} b_{X_2,X'} a_{X',X_1} + \rho_{X_1} a_{X_1,X_2} + \rho_{X_2} a_{X_2,X_1} + r_{X_1,X_2}(s_{X_1} + s_{X_2}), \\
\dot\rho_{X_1} &= \sum_{\{X'\}\cup X_2} r_{X_1,X'}(a_{X',X_1} + a_{X_1,X'}) + 2\rho_{X_1} s_{X_1}.
\end{aligned} \tag{8}$$

Lemma 1 *[6]. For arbitrary tree graph, deformation (8) by virtue of (7) preserves conditions (6) (and, hence, factoring condition (4).*

We now consider arbitrary graph with vertices of valence three and finite number of loops.

Theorem 2 *For arbitrary graph Γ_3 with vertices of valence three and arbitrary number of loops, deformation (8) preserves the periodicity properties iff for any closed cycle with the consequtive (cyclically ordered) vertices $\{X_i\}_{i=1}^n$, $X_{n+1} \equiv X_1$, the two conditions*

$$\text{(a)} \qquad \prod_{k=1}^{n} (-) \frac{q_{X_k,X_{k+1}}}{q_{X_{k+1},X_k}} = 1 \quad \text{(cocycle condition, see [5])}, \tag{9}$$

$$\text{(b)} \qquad \sum_{i=1}^{n} r_{X_i,X_{i+1}} \prod_{k=1}^{i-1} (-) \frac{q_{X_k,X_{k+1}}}{q_{X_{k+2},X_{k+1}}} = 0 \tag{10}$$

are satisfied.

Remark 1 The proof is the lengthy but direct calculation using formulas (7), (8). Note that the time derivative of condition (a) is twice the condition (b), and the only nontrivial part is to prove that the time derivative of condition (b) necessarily vanishes providing both conditions (a) and (b) are satisfied. Therefore, no new conditions arise. Because both condition (a) and condition (b) satisfies the cocycle property, it suffices to impose these conditions only on base cycles of a graph.

References

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